Do corners always scatter?

Eemeli Blåsten*
University of Helsinki

Lassi Päivärinta†
University of Helsinki

John Sylvester‡
University of Washington

November 2012

1 Abstract

We study time harmonic scattering for the Helmholtz equation in $\mathbb{R}^n$. We show that certain penetrable scatterers with rectangular corners scatter every incident wave nontrivially. Even though these scatterers have interior transmission eigenvalues, the relative scattering (a.k.a. far field) operator has a trivial kernel and cokernel at every real wavenumber.

2 Introduction

In an inverse scattering problem one probes an unknown structure with waves and tries to reconstruct the structure from the response. If there is no response to the probing wave, one is tempted to infer that the structure is absent, but this may, or may not be the case. More precisely, we consider the scattering of time-harmonic acoustic waves by an inhomogeneous medium of compact support. The above situation can arise at certain wavenumbers if the relative scattering operator is not injective. Most reconstruction algorithms in inverse scattering theory, such as the linear sampling method of Colton and Kirsch [4], and the factorization method of Kirsch [10], need to avoid such wavenumbers, but do they exist?

In the spherically symmetric case, the existence of non-scattering wavenumbers has been known for a long time [6, 7]. In that case the existence follows from the fact that every real interior transmission eigenvalue implies also the existence of non-scattering waves. On the other hand, the existence of transmission eigenvalues for general positive perturbations of the background was proven in [11]

*Research supported by the Finnish Centre of Excellence in Inverse Problems Research
†Research supported by the ERC 2010 Advanced Grant 267700 and the Finnish Centre of Excellence in Inverse Problems Research
‡Research supported in part by the NSF grant DMS-1007447

and generalized in [3] and [8] to more general situations. Unfortunately this does not answer the original question about the existence of non-scattering waves. The purpose of this paper is to describe a situation where there are infinitely many interior transmission eigenvalues but all incident waves still scatter. We begin with a more detailed description of the problem.

The scattering of time harmonic acoustic waves by a penetrable medium can be modeled by the Helmholtz equation

$$(\Delta + k^2 n^2)u = 0 \quad \text{in } \mathbb{R}^n, \tag{1}$$

where $n(x)$ denotes the index of refraction. In this model, we seek the total wave as

$$u = v^0 + u^+ \tag{2}$$

where v^0 is the *incident wave* and u^+ the outgoing *scattered wave.* This means that

$$(\Delta + k^2)v^0 = 0 \quad \text{in } \mathbb{R}^n \tag{3}$$

and therefore that

$$(\Delta + k^2)u^+ = k^2 m(v^0 + u^+) \tag{4}$$

where the *contrast* m, defined by

$$n^2 = 1 - m, \tag{5}$$

is compactly supported. The relative scattering operator maps the asymptotics of incident waves to the asymptotics of scattered waves. Specifically if $B^s_{p,q}$ denote the Besov spaces [14, 15], any v^0 satisfying (3) that belongs to the Fourier image of the space $B^1_{2,1}$ (which we denote by $\widehat{B^1_{2,1}}$, see Section 5 for this) is a Herglotz wavefunction [1], i.e.

$$v^0(x) = \int_{S^{n-1}} g_0(\theta) e^{ik\theta\cdot x} d\sigma(\theta), \tag{6}$$

for some $g_0 \in L^2(S^{n-1})$ and

$$v^0(r\theta) \sim \frac{e^{ikr}}{(ikr)^{\frac{n-1}{2}}} g_0(\theta) + \frac{e^{-ikr}}{(-ikr)^{\frac{n-1}{2}}} g_0(-\theta). \tag{7}$$

The scattered wave u^+ also has asymptotics

$$u^+(r\theta) \sim \frac{e^{ikr}}{(ikr)^{\frac{n-1}{2}}} \alpha^+(\theta) \tag{8}$$

and the relative scattering operator $S(k)$ maps

$$S(k) : L^2(S^{n-1}) \ni g_0 \mapsto \alpha^+ \in L^2(S^{n-1}). \tag{9}$$

It is a compact and normal operator in $L^2(S^{n-1})$, so never has a bounded inverse, but a number of methods in inverse scattering succeed only if the kernel,

and hence cokernel, of $S(k)$ is trivial. If the contrast $m(x)$ in (4) is compactly supported, then a nontrivial kernel implies that k^2 is an *interior transmission eigenvalue* (ITE) for any domain Ω that contains the interior of the support of m. This means that there is a nontrivial u^+ and v^0 satisfying

$$\begin{aligned}(\Delta + k^2)v^0 &= 0 \quad \text{in } \Omega\\ (\Delta + k^2)u^+ &= k^2 m(v^0 + u^+) \quad \text{in } \Omega\\ u^+|_{\partial\Omega} = 0,\ \frac{\partial u^+}{\partial \nu}&|_{\partial\Omega} = 0\end{aligned} \tag{10}$$

If $m(x)|_\Omega > 0$, then the ITE's are known to be discrete, and if $m(x) > 0$ in all of Ω, there are infinitely many real ITE's [3, 8]. In the spherically symmetric case ($m = m(|x|)$), u^+ extends to all of $\mathbb{R}^n$ as 0, and v^0 extends to $\mathbb{R}^n$ as a spherically harmonic times a Bessel function. In this case, k is called *non-scattering wavenumber*. That is, if there is a nontrivial u^+ and v^0 satisfying

$$\begin{aligned}(\Delta + k^2)v^0 &= 0 \quad \text{in } \mathbb{R}^n\\ (\Delta + k^2)u^+ &= k^2 m(v^0 + u^+) \quad \text{in } \mathbb{R}^n\\ u^+|_{\partial\Omega} = 0,\ \frac{\partial u^+}{\partial \nu}&|_{\partial\Omega} = 0\end{aligned} \tag{11}$$

then k is a *non-scattering wavenumber*, and v^0 a *non-scattering incident wave*. Hence they are wavenumbers for which the relative scattering operator has a non-trivial kernel. In general, the relationship between ITE's and non-scattering frequencies is unclear.

In this paper, we show that, if the contrast $m(x)$ is the characteristic function of an n-dimensional rectangle times a smooth function which is nonzero at at least one corner of the rectangle, then there are no non-scattering frequencies. If, in addition, $m(x) > 0$ on its support, it has infinitely many real ITE's there. For each ITE, the function u^+ satisfying (10) extends to $\mathbb{R}^n$ as 0 but, as we will show, v^0 does not extend, as a solution to (3), to any neighborhood of the nontrivial corner.

3 All Corners Scatter

The main result of this paper is the following

Theorem 3.1. *Suppose that $k \neq 0$, that K is an n-dimensional rectangle, and*

i) $m = \chi_K \varphi(x)$ with $\varphi \in C^\infty(\mathbb{R}^n)$ and $\varphi(x_0) \neq 0$ at at least one corner of K

ii) v^0 satisfies

$$(\Delta + k^2)v^0 = 0 \quad \textit{in } \mathbb{R}^n \tag{12}$$

iii) u^+ satisfies

$$(\Delta + k^2)u^+ = k^2 m(v^0 + u^+) \quad \textit{inside } K \tag{13}$$

$$\textit{and } u^+|_{\partial K} = \frac{\partial u^+}{\partial \nu}|_{\partial K} = 0 \tag{14}$$

then $v^0 = 0$.

Corollary 3.2. *If m satsifies i) and $k \neq 0$, the kernel of the scattering operator is trivial.*

Proof. If the kernel of $S(k)$ is nontrivial, then there is a Herglotz wavefunction satisfying (12) in $\mathbb{R}^n$, and an outgoing u^+ satisfying (13) in $\mathbb{R}^n$ with vanishing far field α^+. Rellich's lemma and unique continuation [5] guarantee that u^+ vanishes outside the support of m. It follows from the fact that $m \in L^\infty$ and $v^0 \in L^2$ that $u^+ \in H^2_{loc}(\mathbb{R}^n)$, and therefore the derivative to ∂K must vanish, so the theorem implies that v_0, and hence, by Fourier inversion in $\mathscr{S}'$, the amplitude g_0 must vanish. □

For the proof of Theorem 3.1 suppose that $u^+ \in H^2_{loc}$ satisfies (13) and (14), v^0 satisfies (12), and $w \in L^2(K)$ satisfies

$$\big(\Delta + k^2(1-m)\big)w = 0 \quad \text{inside } K \tag{15}$$

Then, integration by parts gives

$$\int_K wk^2mv^0 = 0 \tag{16}$$

We will prove the theorem by showing that, for any v^0, and the resulting u^+, we can choose w so that the left hand side of (16) is nonzero.

We devote section 6 to the proof of

Theorem 3.3. *Suppose that $m(x)$ satisfies i) in Theorem 3.1 and $\rho \in \mathbb{C}^n$ satisfies $\rho \cdot \rho = 0$. Then, for $|\rho|$ sufficiently large, there exists w satisfying* (15) *of the form*

$$w = e^{x\cdot\rho}(1+\psi) \tag{17}$$

with

$$\|\psi\|_{L^p(K)} \leq \frac{C}{|\rho|} \tag{18}$$

for any $2 \leq p < \infty$.

We will combine this with the simple lemma

Lemma 3.4. *Suppose that $v^0 \not\equiv 0$ and x_0 is in an open set where $(\Delta + k^2)v^0 = 0$. Then the lowest order homogeneous polynomial in the Taylor series for v^0 at x_0 is harmonic.*

Proof. The function v^0 is real analytic at x_0, so its Taylor expansion doesn't vanish.

$$\begin{aligned} v^0(x) &= P^N(x-x_0) + v^{N+1}(x) \\ \Delta v^0(x) &= \Delta P^N(x-x_0) + \Delta v^{N+1}(x) \\ &= Q^{N-2}(x-x_0) + q^{N-1}(x) \end{aligned} \tag{19}$$

where P^N and Q^{N-2} are homogeneous polynomials of degree N and $N-2$ respectively, and

$$\begin{aligned} \left|v^{N+1}(x)\right| &\leq c\left|x-x_0\right|^{N+1} \\ \left|q^{N-1}(x)\right| &\leq c\left|x-x_0\right|^{N-1}. \end{aligned} \tag{20}$$

We may assume that $N \geq 2$ since otherwise P_N is automatically harmonic. If

$$\Delta v^0 = -k^2 v_0 \tag{21}$$

then

$$\left|Q^{N-2}(x-x_0)\right| = \left|-q^{N-1}(x) - k^2(P^N + v^{N+1})\right| \leq c\left|x-x_0\right|^{N-1}, \tag{22}$$

but Q^{N-2} is homogeneous of order $N-2$, so must be zero. $\square$

In section 7, we will prove

Theorem 3.5. *Let $P^N(x)$ be a nontrivial homogeneous harmonic polynomial on $\mathbb{R}^n$. Then its Laplace transform is given by*

$$\int_{x>0} e^{-x\cdot\rho} P^N(x)dx = Q^{N+n}\left(\tfrac{1}{\rho}\right). \tag{23}$$

where Q^{N+n} is a homogeneous polynomial of degree $N+n$,and $\frac{1}{\rho}$ denotes the vector with j-th component $\frac{1}{\rho_j}$. Moreover, $Q^{N+n}(\frac{1}{\rho})$ does not vanish identically on any open subset of $\rho\cdot\rho = 0$ when $n \geq 3$. In 2D it has a nonzero value in $\{\rho\cdot\rho = 0, \Re\rho > 0\}$.

Returning to the proof of Theorem 3.1, we insert the w from Theorem 3.3 into (16), and expand v^0 as in Lemma 3.4, obtaining

$$0 = \int_K e^{-x\cdot\rho}(1+\psi)m\big(P^N(x) + v^{N+1}(x)\big), \tag{24}$$

which we reorganize as

$$\int_K e^{-x\cdot\rho} m P^N = \int_K e^{-x\cdot\rho} m Q^{N+1}\widetilde{v}^{N+1} + \int_K e^{-x\cdot\rho}\psi m(P^N + Q^{N+1}\widetilde{v}^{N+1}), \tag{25}$$

where we have rewritten $v^{N+1} = -Q^{N+1}\widetilde{v}^{N+1}$ as a homogeneous polynomial times an analytic function $\widetilde{v}^{N+1}$. Note that $\widetilde{v}^{N+1}$ stays bounded in K. Without loss of generality, we will assume that the rectangle is located in the positive orthant $\{x_j > 0\}$, that $x = 0$ is the corner at which m doesn't vanish, and that $m(0) = 1$. The following lemma tells us how the right hand side of (25) decays as $|\rho| \to \infty$.

Lemma 3.6. *Let $R^N(x)$ be a homogeneous polynomial of degree N and $\Re\rho_j > 0$ for all $j = 1, \ldots, n$. Then*

$$\left|\int_{x>0} e^{-x\cdot\rho} R^N(x) f(x) dx\right| \leq C\left|\rho\right|^{-(N+n)+n/p}\left\|f\right\|_{L^p} \tag{26}$$

Proof. Let $\rho = s \cdot \theta$, where $|\theta| = 1$. Then

$$\int_{x>0} e^{-sx\cdot\theta} R^N(x) f(x) dx = \frac{1}{s^{N+n}} \int_{y>0} e^{-y\cdot\theta} R^N(y) f\left(\tfrac{y}{s}\right) dy$$
$$\leq \frac{1}{s^{N+n}} \left\| e^{-y\cdot\theta} R^N(y) \right\|_{L^q} \left\| f\left(\tfrac{y}{s}\right) \right\|_{L^p} = \frac{C_{\theta,n,q,R}}{s^{N+n}} s^{n/p} \left\| f \right\|_{L^p} \quad (27)$$

□

With these preparations we are ready to prove Theorem 3.1. As a consequence of the lemma, (25) becomes

$$\left| \int_K e^{-x\cdot\rho} m P^N \right| \leq \left| \int_K e^{-x\cdot\rho} m Q^{N+1} \widetilde{v}^{N+1} \right| + \left| \int_K e^{-x\cdot\rho} \psi m (P^N + Q^{N+1} \widetilde{v}^{N+1}) \right|$$
$$\leq \frac{C}{|\rho|^{n+N+1}} \left\| m \widetilde{v}^{N+1} \right\|_{L^\infty} + \frac{C}{|\rho|^{n+N-n/p}} \left\| \psi m \widetilde{v}^{N+1} \right\|_{L^p}, \quad (28)$$

which combines with (18) to yield

$$\leq \left\| m v^{N+1} \right\|_{L^\infty} \left(\frac{C}{|\rho|^{n+N+1}} + \frac{C}{|\rho|^{n+N-n/p}} \cdot \frac{C}{|\rho|} \right) \leq \frac{C}{|\rho|^{n+N+(1-n/p)}}. \quad (29)$$

Theorem 3.3 allows us to choose any $2 \leq p < \infty$, say $p = 2n$, so we have

$$\left| \int_{x>0} e^{-x\cdot\rho} m P^N \right| \leq \frac{C}{|\rho|^{n+N+1/2}}. \quad (30)$$

On the other hand, Theorem 3.5 tells us that

$$\left| \int_{x>0} e^{-x\cdot\rho} P^N(x) dx \right| \geq \frac{C}{|\rho|^{N+n}} \quad (31)$$

with C nonzero after a suitable choice of ρ. Because $m(x) - 1$ vanishes at $x = 0$, we have

$$\left| \int_{x>0} e^{-x\cdot\rho} P^N(x) \big(m(x) - 1\big) dx \right| = \left| \int_{x>0} e^{-x\cdot\rho} \widetilde{Q}^{N+1}(x) \widetilde{m}(x) dx \right|$$
$$\leq \frac{C}{|\rho|^{N+n+1}} \left\| \widetilde{m} \right\|_{L^\infty} \quad (32)$$

se we arrive at the contradiction that

$$\frac{C}{|\rho|^{N+n}} \leq \left| \int e^{-x\cdot\rho} P^N(x) dx \right| \leq \frac{C}{|\rho|^{N+n+1/2}} \quad (33)$$

and the theorem is proved. It remains to prove Theorem 3.3 and Theorem 3.5, which we will do in Section 6 and Section 7.

4 Estimates for Fundamental Solutions

In Section 5, we will prove an estimate for a solution to

$$P(D)\psi := (\Delta + \rho \cdot \nabla)\psi = f. \tag{34}$$

We will need to estimate the convolution of the reciprocal $1/P(\xi)$ of the symbol with a Schwartz class function χ, scaled by ε ($\varepsilon = 2^{-j}$ in 6.3). Here the symbol $P(\xi) = -\xi \cdot \xi + i\rho \cdot \xi$ vanishes simply on a codimension 2 sphere in $\mathbb{R}^n$, and the estimates for $\frac{1}{P}$ are essentially the same as the estimates for $\frac{1}{\xi_1 + i\xi_2}$, which vanishes at a point, also a codimension 2 manifold in $\mathbb{R}^2$. We prove a fairly general estimate of this sort. We follow the outline in [13], in order to work in L^p spaces, but our proofs are more geometric, similar to those in [1], which only treats L^2 based spaces ($\widehat{B^{-1}_{2,\infty}}$ and $\widehat{B^{1}_{2,1}}$). Our main theorem is the following:

Theorem 4.1. *Suppose that $\chi(x) \in \mathscr{S}(\mathbb{R}^n, \mathbb{C})$ and $\chi_\varepsilon(x) := \varepsilon^{-n}\chi(\frac{x}{\varepsilon})$. If $P(\xi)$ satisfies*

i) $P : \mathbb{R}^n \to \mathbb{R}^k$ is smooth

ii) $\mathscr{M} = P^{-1}\{0\}$ is compact

iii) $DP_{|\mathscr{M}}$ has constant rank, and

iv) $\liminf_{|\xi|\to\infty} |P| \geq B > 0$

then

a) $\mathscr{M}$ is a smooth embedded codimension k manifold in $\mathbb{R}^n$

b) $\|\chi_\varepsilon * \delta_{\mathscr{M}}\|_{L^\infty} \leq \frac{C}{\varepsilon^k}$

c) If P is real or complex valued ($k = 1$ or 2), then

$$\left\| \chi_\varepsilon * \frac{1}{P} \right\|_{L^\infty} \leq \frac{C}{\varepsilon}. \tag{35}$$

Moreover, if $k \geq 2$ and F is a complex valued function satisfying $|F(P)| \leq \frac{1}{|P|}$ then

$$\|\chi_\varepsilon * F(P)\|_{L^\infty} \leq \frac{C}{\varepsilon}. \tag{36}$$

Remark 4.2. We define

$$\langle \delta_{\mathscr{M}}, \phi \rangle := \int_{\mathscr{M}} \phi \, d\sigma_{\mathscr{M}} \tag{37}$$

where $d\sigma_{\mathscr{M}}$ is the natural element of surface area on $\mathscr{M}$.

Remark 4.3. If $k \geq 2$ then $\frac{1}{P} \in L^1_{loc}$ is a well defined distribution on the whole of $\mathbb{R}^n$. If $k = 1$ we will use the principal value

$$\langle \frac{1}{P}, \phi \rangle := \int_{N_\delta(\mathscr{M})} (\phi(y) - \phi(m(y))) \frac{dy}{P(y)} + \int_{\mathbb{R}^n \setminus N_\delta(\mathscr{M})} \phi(y) \frac{dy}{P(y)}, \tag{38}$$

with $N_\delta(\mathscr{M})$ and $m(y)$ are as defined in the next proposition.

The following proposition recalls some immediate consequences of the implicit function theorem. We don't include a proof.

Proposition 4.4. *Suppose that i), ii) and iii) in Theorem 4.1 are satisfied. Then*

A) $DP|_{\mathscr{M}}$ *has full rank* k

B) $\mathscr{M}$ *is a smooth compact embedded submanifold of* $\mathbb{R}^n$

C) $\exists \delta > 0$ *and a Lipschitz constant* L_δ *such that writing*

$$N_\delta(\mathscr{M}) = \{x \in \mathbb{R}^n \mid d(x, \mathscr{M}) \leq \delta\}, \tag{39}$$

every $x \in N_\delta(\mathscr{M})$ *has a unique closest point* $m(x)$ *in* $\mathscr{M}$. *The map*

$$\eta : N_\delta(\mathscr{M}) \to \mathscr{M} \times B^k_\delta(0) \tag{40}$$

defined by

$$\eta(x) = \left(m(x), |x - m(x)| \frac{DP_{m(x)}(x - m(x))}{\left|DP_{m(x)}(x - m(x))\right|} \right) \tag{41}$$

is a global diffeomorphism from $N_\delta(\mathscr{M})$ *onto* $\mathscr{M} \times B^k_\delta(0)$. *Both* η *and* η^{-1} *are Lipshitz with uniform constant* L_δ.

D) Every point $m \in \mathscr{M}$ *has a* δ*-neighborhood* $U_\delta(m) \subset \mathscr{M}$ *that is diffeomorphic to a ball in* $\mathbb{R}^{n-k}$, *i.e.*

$$\psi_m : U_\delta(m) := B^n_\delta(m) \cap \mathscr{M} \to B^{n-k}_\delta(0). \tag{42}$$

Both ψ_m *and* ψ_m^{-1} *are Lipschitz with uniform constant* L_δ.

Two corollaries (also stated without proof) are:

Corollary 4.5. *For* $x \in \mathbb{R}^n$,

$$\text{Area}\,(B^n_r(x) \cap \mathscr{M}) := \int_{\mathscr{M} \cap B^n_r(x)} d\sigma_{\mathscr{M}} \leq C_\delta r^{n-k} \tag{43}$$

Corollary 4.6. *For* $x \in N_\delta(\mathscr{M})$,

$$|P(x)| \geq C_\delta d(x, \mathscr{M}). \tag{44}$$

We are going to use diffeomorphisms to rewrite integrals over manifolds as integrals over euclidean balls, where we can do some explicit calculations. Since our integrals will involve convolutions with Schwartz class functions, we need to describe the properties that the pullbacks of such functions inherit.

Definition 4.7. A family of ε-mollifiers, $\chi_\varepsilon(x,y)$, defined on $\Omega_1 \times \Omega_2 \subset \mathbb{R}^n \times \mathbb{R}^n$ satisfies

i) $\sup_{x\in\Omega_1} \int_{\Omega_2} |\chi_\varepsilon(x,y)|\, dy \le C$

ii) $|\chi_\varepsilon(x,y)| \le \frac{C_N}{\varepsilon^n}\left(\frac{\varepsilon}{|x-y|}\right)^N$ for all $N \in \mathbb{N}$

iii) $|\nabla_y \chi_\varepsilon(x,y)| \le \frac{C_N}{\varepsilon^{n+1}}\left(\frac{\varepsilon}{|x-y|}\right)^N$ for all $N \in \mathbb{N}$

Lemma 4.8. *If $\chi \in \mathscr{S}$, then*

$$\chi_\varepsilon(x,y) := \frac{1}{\varepsilon^n}\chi\left(\frac{x-y}{\varepsilon}\right) \tag{45}$$

is a family of ε-mollifiers defined on $\Omega_1 \times \Omega_2 = \mathbb{R}^n \times \mathbb{R}^n$.

Definition 4.9. The pullback of a family of ε-mollifiers is defined[1] to be

$$\psi^*\chi_\varepsilon(x,y) := \chi_\varepsilon(\psi(x),\psi(y)). \tag{46}$$

The next lemma explains why we need to work with general ε-mollifiers.

Lemma 4.10. *If ψ and ψ^{-1} are uniformly Lipschitz diffeomorphisms, then the pullback of a family of ε-mollifiers is a family of ε-mollifiers.*

Proof. Let L_1 and L_2 be the Lipschitz constants for ψ and ψ^{-1}, respectively. For i), we estimate

$$\begin{aligned}\sup_{x\in\psi^{-1}(\Omega_1)} \int_{\psi^{-1}(\Omega_2)} \chi_\varepsilon(\psi(x),\psi(y))dy &= \sup_{x\in\Omega_1}\int_{\Omega_2} \chi_\varepsilon(x,y)\frac{dy}{\det(D\psi(y))} \\ &\le \sup_{x\in\Omega_1} L_2^n \int_{\Omega_2}\chi_\varepsilon(x,y)dy \end{aligned} \tag{47}$$

Next

$$|\chi_\varepsilon(\psi(x),\psi(y))| \le \frac{C_N}{\varepsilon^n}\left(\frac{\varepsilon}{|\psi(x)-\psi(y)|}\right)^N \le \frac{C_N L_2^N}{\varepsilon^n}\left(\frac{\varepsilon}{|x-y|}\right)^N. \tag{48}$$

Finally, for iii),

$$\begin{aligned}|\nabla_y \chi_\varepsilon(\psi(x),\psi(y))| &= |D\psi \cdot \nabla_v \chi_\varepsilon(u,v)|\Big|_{\substack{u=\psi(x)\\ v=\psi(y)}} \\ &\le L_1\frac{C_N}{\varepsilon^{n+1}}\left(\frac{\varepsilon}{|\psi(x)-\psi(y)|}\right)^N \le \frac{C_N L_1 L_2^N}{\varepsilon^{n+1}}\left(\frac{\varepsilon}{|x-y|}\right)^N\end{aligned} \tag{49}$$

□

[1]It seems natural to include a factor of $\det(D\psi)$ in (46), to treat $\chi_\varepsilon dx_1 \wedge \cdots \wedge dx_n$ as an n-form. We don't add the factor because it makes the proof of Lemma 4.10 slightly longer.

Proposition 4.11. *Let χ_ε be a family of ε-mollifiers defined on $\Omega_1 \times \Omega_2 \subset \mathbb{R}^n \times \mathbb{R}^n$ and $\mathscr{M}$ a compact embedded submanifold of $\mathbb{R}^n$ of codimension k. Then*

$$\sup_{x\in\Omega_1} \int_{\mathscr{M}\cap\Omega_2} |\chi_\varepsilon(x,m) d\sigma_{\mathscr{M}}(m)| \leq \frac{C}{\varepsilon^k} \tag{50}$$

for small ε.

Proof. We may assume that $\mathscr{M} \subset \Omega_2$. Let δ be the uniform constant in Proposition (4.4). Fix $x \in \Omega_1$ and assume that $\varepsilon < \delta$. According to ii) in Definition 4.7 we have

$$\left| \int_{\mathscr{M}\cap\{m||x-m|\geq\delta\}} \chi_\varepsilon(x,m) d\sigma_{\mathscr{M}}(m) \right| \leq \frac{C_N}{\varepsilon^n} \left(\frac{\varepsilon}{\delta}\right)^N \operatorname{area}(\mathscr{M}). \tag{51}$$

On the other hand

$$\begin{aligned} \left| \int_{\mathscr{M}\cap\{m||x-m|\leq\varepsilon\}} \chi_\varepsilon(x,m) d\sigma_{\mathscr{M}}(m) \right| &\leq \frac{C_0}{\varepsilon^n} \operatorname{area}\left(\mathscr{M} \cap B^n_\varepsilon(x)\right) \\ &\leq \frac{C_0}{\varepsilon^n} L_\delta^{n-k} \varepsilon^{n-k} = \frac{C_0 L_\delta^{n-k}}{\varepsilon^k}, \end{aligned} \tag{52}$$

where L_δ is the Lipschitz constant. To estimate the remaining part of the integral, we use local coordinates ψ, based at $m(x)$, the point on $\mathscr{M}$ closest to x, as described in Proposition 4.4 D). Let $\Psi = \psi^{-1}$. Then

$$\left| \int_{\mathscr{M}\cap\{m|\varepsilon<|x-m|<\delta\}} \chi_\varepsilon(x,m) d\sigma_{\mathscr{M}}(m) \right| = \left| \int_{B_\delta^{n-k}(0)\setminus B_\varepsilon^{n-k}(0)} \Psi^*\chi_\varepsilon \Psi^* d\sigma_{\mathscr{M}} \right| \tag{53}$$

Because

$$\begin{aligned} |\chi_\varepsilon(x,m)| &\leq \frac{C_N}{\varepsilon^n} \left(\frac{\varepsilon}{(|x-m(x)|^2 + |m(x)-m|^2)^{1/2}} \right)^N \\ &\leq \frac{C_N}{\varepsilon^n} \left(\frac{\varepsilon}{|m(x)-m|} \right)^N = \frac{C_N}{\varepsilon^n} \left(\frac{\varepsilon}{\rho} \right)^N, \end{aligned} \tag{54}$$

where $\rho = |m(x) - m|$, we may use polar coordinates centered at $m(x)$ to see that

$$\begin{aligned} &\int_{B_\delta^{n-k}(0)\setminus B_\varepsilon^{n-k}(0)} |\Psi^*\chi_\varepsilon \Psi^* d\sigma_{\mathscr{M}}| \leq L_\delta^{n-k} \int_{S^{n-k-1}} d\sigma_{S^{n-k-1}} \int_\varepsilon^\delta \frac{C_N}{\varepsilon^n} \left(\frac{\varepsilon}{\rho}\right)^N \rho^{n-k-1} d\rho \\ &\leq L_\delta^{n-k} \omega_{n-k-1} C_N \varepsilon^{N-n} \frac{|\delta^{n-k-N} - \varepsilon^{n-k-N}|}{|n-k-N|} \leq L_\delta^{n-k} \omega_{n-k-1} C_N \frac{\varepsilon^{-k}}{|n-k-N|} \end{aligned} \tag{55}$$

where S_{n-k-1} is the unit sphere in $\mathbb{R}^{n-k}$ and ω_{n-k-1} its surface measure. The claim follows by taking $N > n - k$. ∎

Remark 4.12. In the proof of Theorem 4.11, when considering $x \in N_\delta(\mathscr{M})$, we only required that the mollifier satisfy

$$|\chi_\varepsilon(x,y)| \leq \frac{C_N}{\varepsilon^n}\left(\frac{\varepsilon}{|m(x)-m(y)|}\right)^N. \tag{56}$$

We will use this observation in the proof of Theorem 4.13

The following theorem finishes the proof of Theorem 4.1. Note that unlike the claim in Proposition 4.11, the estimate here is independent of k.

Theorem 4.13. *Let χ_ε be a family of ε-mollifiers, ε small enough, $\mathscr{M}$, P and $k \geq 2$ satisfy the conditions in Theorem 4.1, and $F : \mathbb{R}^k \to \mathbb{C}$ such that $|F(P)| \leq \frac{C}{|P|}$. Then*

$$\left|\int_{\mathbb{R}^n} \chi_\varepsilon(x,y)F(P(y))dy\right| \leq \frac{C}{\varepsilon}. \tag{57}$$

If $k = 1$ then

$$\left|\int_{\mathbb{R}^n} \frac{\chi_\varepsilon(x,y)}{P(y)}dy\right| \leq \frac{C}{\varepsilon}, \tag{58}$$

where $\frac{1}{P}$ is defined by principal value as in Remark 4.3.

Proof. We assume that $\varepsilon < \frac{\delta}{2}$, with δ the constant in Proposition 4.4 C). Because $|F(P)| \leq \frac{C}{|P|} \leq \frac{C_\delta}{\varepsilon}$ on $N_\delta(\mathscr{M}) \setminus N_\varepsilon(\mathscr{M})$ and $\leq C_\delta$ outside $N_\delta(\mathscr{M})$,

$$\int_{\mathbb{R}^n\setminus N_\varepsilon(\mathscr{M})} |\chi_\varepsilon F(P)|\, dy \leq \sup_{y\in\mathbb{R}^n\setminus N_\varepsilon(\mathscr{M})} |F(P(y))|\, \|\chi_\varepsilon\|_{L^1} \leq \frac{C}{\varepsilon}. \tag{59}$$

For the moment, we restrict to the case that $k = \operatorname{codim}(\mathscr{M}) \geq 2$, so that $F(P) \in L^1(\mathbb{R}^n)$. If $x \notin N_\delta(\mathscr{M})$, then

$$\sup_{y\in N_\epsilon(\mathscr{M})} |\chi_\varepsilon(x,y)| \leq \left(\frac{\varepsilon}{\delta/2}\right)^N \frac{C_N}{\varepsilon^n} \tag{60}$$

so that

$$\sup_{x\notin N_\delta(\mathscr{M})} \int_{N_\varepsilon(\mathscr{M})} |\chi_\varepsilon F(P)|\, dy \leq \int_{N_\varepsilon(\mathscr{M})} |F(P)|\, dy \left(\frac{\varepsilon}{\delta/2}\right)^N \frac{C_N}{\varepsilon^n}. \tag{61}$$

and choosing $N \geq n-1$ shows that this bounded by a constant over ε.

If $x \in N_\delta(\mathscr{M})$, we can use the diffeomorphism η and its inverse H, described in C) of Proposition 4.4 to obtain

$$\begin{aligned}\sup_{x\in N_\delta(\mathscr{M})} &\int_{N_\varepsilon(\mathscr{M})} |\chi_\varepsilon(x,y)F(P(y))|\, dy \\ &= \sup_{u\in \mathscr{M}\times B^k_\delta(0)} \int_{\mathscr{M}\times B^k_\varepsilon(0)} |H^*\chi_\varepsilon(u,v)F(P(H(v)))| \frac{d\sigma_{\mathscr{M}}(m)ds}{|\det(D\eta)|} \end{aligned} \tag{62}$$

where $v = (m,s) \in \mathscr{M} \times B^k_\varepsilon(0)$. Because $|F(P(H(s)))| \leq \frac{C}{|P(y)|} \leq \frac{C}{|s|}$ here and $|\det(D\eta)|$ is bounded from below by the the n-th power of the Lipschitz constant L_2, this is bounded by

$$\leq CL_2^{-n} \int_{B^k_\varepsilon(0)} \left(\sup_{u \in \mathscr{M} \times B^k_\delta(0)} \int_{\mathscr{M}} |H^* \chi_\varepsilon| \, d\sigma_{\mathscr{M}} \right) \frac{1}{|s|} ds. \tag{63}$$

For each fixed s,

$$|H^* \chi_\varepsilon| \leq \frac{C_N}{\varepsilon^n} \left(\frac{\varepsilon}{|u - (m,s)|} \right)^N, \tag{64}$$

so according to Remark 4.12 we can apply Proposition 4.11 to the manifold $\mathscr{M} \times \{s\}$ to show that the quantity in brackets in (63) satisfies

$$\sup_{u \in \mathscr{M} \times B^k_\delta(0)} \int_{\mathscr{M}} |H^* \chi_\varepsilon| \, d\sigma_{\mathscr{M}} \leq \frac{C}{\varepsilon^k}. \tag{65}$$

This implies the estimate

$$\sup_{x \in N_\delta(\mathscr{M})} \int_{N_\varepsilon(\mathscr{M})} |\chi_\varepsilon(x,y) F(P(y))| \, dy \leq \int_{B^k_\varepsilon(0)} \frac{C}{\varepsilon^k} \frac{ds}{|s|} = \frac{C}{\varepsilon^k} \cdot \varepsilon^{k-1}, \tag{66}$$

which completes the proof in the codimension 2 case.

If $\mathscr{M}$ is of codimension one we have the definition

$$\langle \frac{1}{P}, \phi \rangle = \int_{N_\delta(\mathscr{M})} (\phi(y) - \phi(m(y))) \frac{dy}{P(y)} + \int_{\mathbb{R}^n \setminus N_\delta(\mathscr{M})} \phi(y) \frac{dy}{P(y)} \tag{67}$$

and note that this agrees with $\int_{\mathbb{R}^n} \phi \frac{dy}{P}$ for all $\phi \in C_0^\infty(\mathbb{R}^n \setminus \mathscr{M})$. With this definition,

$$\frac{1}{P} * \chi_\varepsilon = \int_{\mathbb{R}^n \setminus N_\varepsilon} \chi_\varepsilon \frac{dy}{P} + \int_{N_\varepsilon(\mathscr{M})} \frac{(\chi_\varepsilon(x,y) - \chi_\varepsilon(x, m(y)))}{P(y)} dy. \tag{68}$$

We estimate the first integral as we did in the codimension ≥ 2 case, and rewrite the second as

$$\int_{\mathscr{M}} \left[\int_{-\varepsilon}^{\varepsilon} \frac{\chi_\varepsilon(m(x), \nu(x), m(y), \nu(y)) - \chi_\varepsilon(m(x), \nu(x), m(y), 0)}{P(m(y), \nu, y)} d\nu(y) \right] d\sigma_{\mathscr{M}}. \tag{69}$$

where $m(x)$ again denotes the closest point on $\mathscr{M}$, and $\nu(x)$ are the normal coordinates, given explicitly by the second component on the right hand side of equation (41). If we call the integral in brackets $\widetilde{\chi_\varepsilon}$, we see that

$$|\widetilde{\chi_\varepsilon}| \leq \frac{C_N}{\varepsilon^n} \left| \frac{\varepsilon}{|m(x) - m(y)|} \right|^N \tag{70}$$

so that Remark 4.12 applies here, and we may conclude that $\left| \int_{\mathscr{M}} \widetilde{\chi_\varepsilon} d\sigma_{\mathscr{M}} \right| \leq \frac{C}{\varepsilon^k}$ with $k = 1$ in this case. $\square$

5 Function spaces

We will prove new estimates for $G_\rho = \mathscr{F}^{-1}(-\xi\cdot\xi + i\rho\cdot\xi)^{-1}\mathscr{F}$ in spaces whose Fourier-multipliers are L^∞ functions. In particular, the spaces we are interested in are the space $\widehat{L^p}(K)$ of distributions supported in the rectangle K whose Fourier-transform is in L^p, and the translation-invariant space $\widehat{B^s_{p,q}}(\mathbb{R}^n)$ which consists of the Fourier-transforms of Besov distributions $B^s_{p,q}$.

We will first start with the definition and basic properties of $\widehat{L^p}$.

Definition 5.1. Let $0 < p \leq \infty$ and $X \subset \mathbb{R}^n$. Then

$$\widehat{L^p}(X) := \{f \in \mathscr{S}'(\mathbb{R}^n) \mid \operatorname{supp} f \subset X, \widehat{f} \in L^p(\mathbb{R}^n)\}, \tag{71}$$

with norm $\|f\|_{\widehat{L^p}} := \left\|\widehat{f}\right\|_p$.

Proposition 5.2. *Let $X \subset \mathbb{R}^n$ be closed. Then $\widehat{L^p}(X)$ is a Banach space for $1 \leq p \leq \infty$. If X is compact, then increasing p makes the space larger.*

Proof. The first claim follows from the facts that $L^p(\mathbb{R}^n)$ is a Banach space and that convergence in L^p implies convergence in $\mathscr{S}'$, which shows that the limit distribution is also supported in X.

Take $\phi \in C_0^\infty$ such that $\phi_{|X} \equiv 1$ and $0 < p < q \leq \infty$. Then by Young's inequality

$$\|f\|_{\widehat{L^q}} = \left\|\widehat{\phi} * \widehat{f}\right\|_{L^q} \leq C_{p,q}\left\|\widehat{\phi}\right\|_{L^r}\|f\|_{\widehat{L^p}} \tag{72}$$

for $\frac{1}{p} + \frac{1}{r} = \frac{1}{q} + 1$. See also [15, 1.4.1]. ☐

Remark 5.3. $\widehat{L^p}(X)$ is a quasi-Banach space for $0 < p < 1$.

Lemma 5.4. *Let $1 < p < \infty$ and $f \in \widehat{L^p}(\mathbb{R}^n)$. Let $k < n$ and assume that* $\operatorname{supp} f \subset \cup_{j=1}^N F_j$ *where each F_j is a k-dimensional hyperplane. Then $f = 0$.*

Proof. Consider the case $k = 0$ first and let $p_1, \ldots, p_N$ be points. The distribution f is compactly supported so has finite order m and so (cf. [9, 2.3.4])

$$f = \sum_{j=1}^N \sum_{|\alpha|\leq m} c_\alpha \partial^\alpha \delta_{p_j}. \tag{73}$$

Hence

$$L^p(\mathbb{R}^n) \ni \widehat{f}(\xi) = \sum_{j=1}^N \sum_{|\alpha|\leq m} c'_\alpha \xi^\alpha, \tag{74}$$

which is possible only for $f = 0$.

Assume that the claim is true for some $0 \leq k \leq n-2$ and that f is supported on a finite union of $k+1$-dimensional hyperplanes F_j. Let

$$S_k = \cup\{F_j \cap F_l \mid F_j \neq F_l\} = \bigcup_{j=1}^N \bigcup_{l\neq j} F_j \cap F_l \tag{75}$$

be the set of points belonging to the intersections of at least two $k+1$-dimensional planes. It is a finite union of k-planes. We will show that $\operatorname{supp} f \subset S_k$.

Let $\varphi \in C_0^\infty(\mathbb{R}^n \setminus S_k)$ and $\epsilon = \operatorname{dist}(S_k, \operatorname{supp}\varphi)$. Now, for each j, take an open set Ω_j such that

1) $\Omega_j \supset (\operatorname{supp}\varphi \cap F_j) \setminus B(S_k, \epsilon)$,

2) $\cup\Omega_j$ does not intersect $B(S_k, \frac{\epsilon}{2})$,

3) $\Omega_j \cap \Omega_l \cap \operatorname{supp}\varphi = \emptyset$ for $j \neq l$,

4) $\cup\Omega_j \supset \operatorname{supp}\varphi$.

Let $\psi_j \in C_0^\infty(\Omega_j)$ be a partition of unity such that $\sum \psi_j \equiv 1$ on $\operatorname{supp}\varphi$. Now

$$\langle f, \varphi\rangle = \sum_{j=1}^{N} \langle f\psi_j, \varphi\rangle, \tag{76}$$

each $f\psi_j \in \widehat{L^p}(\mathbb{R}^n)$ and $\operatorname{supp} f\psi_j \subset F_j$. Rotating and translating won't change the norm of $f\psi_j$, so we may assume that $\operatorname{supp} f\psi_j \subset \{0\} \times \mathbb{R}^{n-1}$.

Now

$$\widehat{f\psi_j}(\xi) = \langle f\psi_j, e^{ix_1\xi_1} e^{ix'\cdot\xi'}\rangle = \langle g_{\xi'}, e^{ix_1\xi_1}\rangle = \widehat{g_{\xi'}}(\xi_1), \tag{77}$$

where $g_{\xi'} \in \mathscr{D}'(\mathbb{R})$ is given by $\langle g_{\xi'}, \phi\rangle = \langle f\psi_j, \phi(x)e^{ix'\cdot\xi'}\rangle$. Fubini's theorem and (77) show that $g_{\xi'} \in \widehat{L^p}(\mathbb{R})$ for almost all ξ'. Moreover $\operatorname{supp} g_{\xi'} \subset \{0\}$. Hence the first part of the proof implies that g, and so $f\psi_j$ is zero. Thus $\langle f, \varphi\rangle = 0$, and so $\operatorname{supp} f \subset S_k$. The induction assumption implies that $f = 0$. $\square$

We will need a dyadic partition of unity for two reasons. Firstly to be able to estimate the Faddeev operator G_ρ and secondly to define the spaces $\widehat{B^s_{p,q}}$. To introduce the notation let

$$1 = \Phi_0(s) + \sum_{j=1}^{\infty} \Phi_j(s) \tag{78}$$

where Φ_0 and Φ are C^∞ even functions of $s \in \mathbb{R}$, and

$$\operatorname{supp}\Phi_0 \subset [-1, 1] \tag{79}$$

$$\operatorname{supp}\Phi \subset]\tfrac{1}{2}, 2[\tag{80}$$

$$\Phi_j(s) := \Phi(\tfrac{s}{2^j}) \quad \text{for } j > 1 \tag{81}$$

$$R_j := 2^j \tag{82}$$

Finally for $x \in \mathbb{R}^n$

$$\phi_j(x) := \Phi_j(|x|). \tag{83}$$

For a tempered distribution f we let

$$f_j := \phi_j f. \tag{84}$$

Recall [2, 14, 15] that for $s \in \mathbb{R}$, $0 < p \le \infty$ and $0 < q \le \infty$ the Besov-space is defined by

$$B^s_{p,q} = B^s_{p,q}(\mathbb{R}^n) = \{f \in \mathscr{S}'(\mathbb{R}^n) \mid \|f\|_{B^s_{p,q}} < \infty\} \tag{85}$$

where

$$\|f\|_{B^s_{p,q}} := \left(\sum_{j=0}^{\infty} \left(R_j^s \left\| \mathscr{F}^{-1}(\phi_j \widehat{f}) \right\|_{L^p} \right)^q \right)^{1/q} \tag{86}$$

with the usual modification for $q = \infty$.

Definition 5.5. We say that $f \in \widehat{B^s_{p,q}}$ if $\widehat{f} \in B^s_{p,q}$ and write $\|f\|_{\widehat{B^s_{p,q}}} = \|\widehat{f}\|_{B^s_{p,q}}$. Note that

$$\|f\|_{\widehat{B^s_{p,q}}} = \left(\sum_{j=0}^{\infty} \left(R_j^s \left\| \widehat{f}_j \right\|_{L^p} \right)^q \right)^{1/q} = \left(\sum_{j=0}^{\infty} \left(R_j^s \left\| \widehat{\phi_j} * \widehat{f} \right\|_{L^p} \right)^q \right)^{1/q}. \tag{87}$$

We will use the norms

$$\|f\|_{\widehat{B^{-1}_{p,\infty}}} := \sup_{0 \le j < \infty} \tfrac{1}{R_j} \left\| \widehat{f}_j \right\|_p \tag{88}$$

$$\|f\|_{\widehat{B^{1}_{p,1}}} := \sum_{j=0}^{\infty} R_j \left\| \widehat{f}_j \right\|_p \tag{89}$$

In the case $p = 2$ these two norms were used in [1] to study constant coefficient PDE's with simple characteristics, including, as the principal example, the free Helmholtz equation. The authors showed, in particular, that the incident waves in $\widehat{B^{-1}_{2,\infty}}$ were exactly the Herglotz wave functions.

Proposition 5.6. *Let $s \in \mathbb{R}$, $1 \le p, q \le \infty$. Then $\widehat{B^s_{p,q}}$ is a Banach space.*

Proof. It follows from the result for Besov spaces. See [15, 2.3.3]. $\square$

6 Estimates for the Faddeev operator

In this section we prove Theorem 3.3, that is, we show the existence of CGO solutions $e^{x\cdot\rho}(1+\psi)$ with the norm estimate $\|\psi\|_{L^p(K)} \le \frac{C}{|\rho|}$ holding for all $2 \le p < \infty$. We begin by noting that if $\rho \cdot \rho = 0$, then

$$w = e^{x\cdot\rho}(1+\psi) \tag{90}$$

satisfies $(\Delta - Q)w = 0$ if and only if ψ satisfies

$$(\Delta + 2\rho \cdot \nabla)\psi = Q(1+\psi). \tag{91}$$

In the case of acoustic scattering of (15) we have $Q = -k^2(1-m)$ inside $\operatorname{supp} m$, but we do not make that assumption now. Our main task is to estimate solutions of

$$(\Delta + \rho \cdot \nabla)u = f. \tag{92}$$

Because $g_\rho(\xi) = (-\xi \cdot \xi + i\rho \cdot \xi)^{-1} \in L^1_{loc}(\mathbb{R}^n)$ the operator

$$G_\rho f := \mathscr{F}^{-1}\left(\frac{\widehat{f}(\xi)}{-\xi \cdot \xi + i\rho \cdot \xi}\right) \tag{93}$$

is well-defined on any compactly supported distribution f and is therefore a right inverse for $\Delta + \rho \cdot \nabla$. Then formally

$$\psi = \sum_{k=1}^{\infty} (G_\rho Q)^k 1 \tag{94}$$

will solve (91).

We will split the kernel $\mathscr{K} = \mathscr{F}^{-1}(-\xi\cdot\xi + i\rho\cdot\xi)^{-1}$ on the x-side and estimate the pieces on the Fourier-side.

Lemma 6.1. *Let $n \geq 2$, ϕ_j be the dyadic partition of unity of* (83) *and $\rho\cdot\rho = 0$. Then*

$$\|\mathscr{F}(\phi_j \mathscr{K})\|_\infty \leq \frac{C2^j}{|\rho|}, \tag{95}$$

where C does not depend on $|\rho|$ or j.

Proof. We will only prove the case $j \geq 1$. The case $j = 0$ follows similarly. Let $\rho = r\theta$, $r = |\rho|$. Set $\eta = r\tau$ and $\xi = r\zeta$. Then, since $\phi_j(x) = \phi(2^{-j}x)$,

$$\begin{aligned}
\|\mathscr{F}(\phi_j \mathscr{K})\|_\infty &= \sup_{\eta \in \mathbb{R}^n} \left| \int 2^{jn} \widehat{\phi}\left(2^j(\eta - \xi)\right) \frac{1}{\xi \cdot \xi - ir(\theta \cdot \xi)} d\xi \right| \\
&= \sup_{\tau \in \mathbb{R}^n} \left| \int r^n 2^{jn} \widehat{\phi}\left(r2^j(\tau - \zeta)\right) \frac{1}{r^2(\zeta \cdot \zeta - i\theta \cdot \zeta)} d\zeta \right| \\
&= \frac{1}{r^2} \sup_{\tau \in \mathbb{R}^n} \left| \int (r2^j)^n \widehat{\phi}\left(r2^j(\tau - \zeta)\right) \frac{1}{\zeta \cdot \zeta - i\theta \cdot \zeta} d\zeta \right|.
\end{aligned} \tag{96}$$

Let $\chi = \widehat{\phi}$, $\varepsilon = (r2^j)^{-1}$, $F(z) = \frac{1}{z}$, and $P(\zeta) = \zeta\cdot\zeta - i\theta\cdot\zeta$. The equation $\theta\cdot\theta = 0$ implies the assumptions of Theorem 4.13. Applying it proves the claim. □

Corollary 6.2. *Let ϕ_j be the dyadic partition of unity of* (83)*, $\rho \cdot \rho = 0$ and $Q \subset \mathbb{R}^n$ be closed. Then*

$$f \mapsto (\phi_j \mathscr{K}) * f : \widehat{L^p}(Q) \to \widehat{L^p}\left(Q + \operatorname{supp} \phi_j\right). \tag{97}$$

and

$$\|(\phi_j \mathscr{K}) * f\|_{\widehat{L^p}(Q + \operatorname{supp} \phi_j)} \leq \frac{C2^j}{|\rho|} \|f\|_{\widehat{L^p}(Q)}. \tag{98}$$

Proof. The claim for the support follows from convolution's basic properties and the norm estimate follows from Lemma 6.1. $\square$

Proposition 6.3. *Let $s \in \mathbb{R}$, $0 < p, q \leq \infty$, $\rho \cdot \rho = 0$ and $Q \subset \mathbb{R}^n$ be compact. Then*

$$G_\rho : \widehat{L^p}(Q) \to \widehat{B^{-1}_{p,\infty}} \tag{99}$$

$$G_\rho : \widehat{B^{1}_{p,1}} \to \widehat{B^{-1}_{p,\infty}} \tag{100}$$

with operator norm at most $\frac{C}{|\rho|}$ in both cases.

Proof. Let $f \in \widehat{L^p}(Q)$ and ϕ_j be the dyadic partition from (83). Denote the convolution kernel of G_ρ by $\mathscr{K}$. Now

$$\phi_k((\phi_j \mathscr{K}) * f) : \widehat{L^p}(Q) \to \widehat{L^p}((Q + \operatorname{supp} \phi_j) \cap \operatorname{supp} \phi_k) \tag{101}$$

by Corollary 6.2. Assume the integer d_Q is so large that $Q \subset B(0, 2^{d_K - 1})$. We have

$$\operatorname{supp} \phi_j + Q \subset B(0, 2^{j+1} + 2^{d_Q - 1}) \setminus B(0, 2^{j-1} - 2^{d_Q - 1}) \tag{102}$$

Comparing the supports we see that $\phi_k(\phi_j \mathscr{K} * f) \equiv 0$ for $j > 1 + k + d_Q$. Hence

$$\phi_k G_\rho f = \phi_k\Big(\sum_{j=0}^{\infty} \phi_j \mathscr{K} * f\Big) = \phi_k\Big(\sum_{j=0}^{1+k+d_Q} \phi_j \mathscr{K} * f\Big), \tag{103}$$

so

$$\begin{aligned}
\|\mathscr{F}(\phi_k G_\rho f)\|_p &\leq \sum_{j=0}^{1+k+d_Q} \left\| \widehat{\phi_k} * ((\widehat{\phi_j} * \widehat{\mathscr{K}}) \widehat{f}) \right\|_p \\
&\leq \sum_{j=0}^{1+k+d_Q} \left\| \widehat{\phi_k} \right\|_1 \left\| \widehat{\phi_j} * \widehat{\mathscr{K}} \right\|_\infty \left\| \widehat{f} \right\|_p = \sum_{j=0}^{1+k+d_Q} \left\| \widehat{\phi} \right\|_1 \frac{C 2^j}{|\rho|} \|f\|_{\widehat{L^p}} \\
&\leq C \|\phi\|_{\widehat{L^1}} \frac{2^{2+k+d_Q} - 1}{|\rho|} \|f\|_{\widehat{L^p}},
\end{aligned} \tag{104}$$

where C is independent of k and $|\rho|$. Hence

$$\|G_\rho f\|_{\widehat{B^{-1}_{p,\infty}}} = \sup_k 2^{-k} \|\mathscr{F}(\phi_k G_\rho f)\|_p \leq \frac{C}{|\rho|} \|f\|_{\widehat{L^p}(Q)}. \tag{105}$$

To prove (100), we do not assume anything about the support of f, but use the pieces $\phi_m f$ instead. Note that $\operatorname{supp} \phi_m f \subset B(0, 2^{m+1})$, so $d_Q = m + 2$. According to (104)

$$\|\mathscr{F}(\phi_k G_\rho(\phi_m f))\|_p \leq C \|\phi\|_{\widehat{L^1}} \frac{2^{k+m}}{|\rho|} \left\| \widehat{\phi_m f} \right\|_p \tag{106}$$

This implies the desired estimate since

$$\|G_\rho f\|_{\widehat{B^{-1}_{p,\infty}}} \le \sup_k 2^{-k} \|\mathscr{F}(\phi_k G_\rho f)\|_p \le \sup_k 2^{-k} \sum_{m=0}^{\infty} \|\mathscr{F}(\phi_k G_\rho(\phi_m f))\|_p$$

$$\le \sup_k \sum_{m=0}^{\infty} C \|\phi\|_{\widehat{L^1}} \frac{2^m}{|\rho|} \left\|\widehat{\phi_m f}\right\|_p = \frac{C\|\phi\|_{\widehat{L^1}}}{|\rho|} \|f\|_{\widehat{B^1_{p,1}}}. \quad (107)$$

□

We will need a fact about the Hilbert transform. This will be used to show the mapping properties of the multiplication operator m where $m = \chi_K \varphi$ as in Theorem 3.1.

Lemma 6.4. *Let χ_K be the characteristic function of the n-cube $K = [0,1]^n$ and $1 < p < \infty$. Then*

$$f \mapsto \widehat{\chi_K} * f : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n) \quad (108)$$

and $\chi_R \in \widehat{L^p}(K)$.

Proof. It is enough to prove the one dimensional case since χ_K can be separated into n characteristic functions of an interval. Then use Fubini.

First note that

$$\chi_{[0,1]}(x) = \frac{\operatorname{sgn}(x) - \operatorname{sgn}(x-1)}{2}, \quad (109)$$

which implies

$$\widehat{\chi_{[0,1]}}(\xi) = \frac{i}{\sqrt{2\pi}}\big(-p.v.\tfrac{1}{\xi} + e^{-i\xi}p.v.\tfrac{1}{\xi}\big). \quad (110)$$

So $\left|\widehat{\chi_{[0,1]}}(\xi)\right| \le \min(1, |\xi|^{-1}) \in L^p(\mathbb{R})$ for $1 < p \le \infty$. The equation also gives

$$\widehat{\chi_{[0,1]}} * g(x) = i\sqrt{\frac{\pi}{2}}\Big(-\mathscr{H}g(x) + e^{-ix}\mathscr{H}\big(e^{i\cdot}g\big)(x)\Big), \quad (111)$$

where the Hilbert transform maps $\mathscr{H} : L^p(\mathbb{R}) \to L^p(\mathbb{R})$. Hence $\widehat{\chi_K} \in L^p(\mathbb{R}^n)$ and $\widehat{\chi_K} * : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$. □

The results of the following corollary are natural because Fourier transforms of compactly supported distributions are analytic, so infinitely smooth. Reducing the support does not necessarily change the smoothness on the x-side, so we have to keep p fixed.

Corollary 6.5. *Let χ_K be as above and $1 < p < \infty$. Then the mapping $f \mapsto \chi_K f := (2\pi)^{-n/2}\mathscr{F}^{-1}(\widehat{\chi_K} * \widehat{f})$ is well defined for $f \in \widehat{L^p}(\mathbb{R}^n) \cup \widehat{B^s_{p,q}}$ and has the following properties*

$$\chi_R \cdot : \widehat{L^p}(\mathbb{R}^n) \to \widehat{L^p}(K) \quad (112)$$

$$\chi_R \cdot \ \textit{is identity on } \widehat{L^p}(K) \quad (113)$$

$$\chi_R \cdot : \widehat{B^s_{p,q}} \to \widehat{B^t_{p,r}} \cap \widehat{L^p}(K) \quad (114)$$

for any $t, s \in \mathbb{R}$ and $0 < q, r \le \infty$.

Proof. The operator is well defined $\widehat{L^p}(\mathbb{R}^n) \to \widehat{L^p}(\mathbb{R}^n)$ by lemma 6.4. The claim for the support follows by approximating $\widehat{f}$ with a smooth test function and writing the definitions out. Thus it maps $\widehat{L^p}(\mathbb{R}^n) \to \widehat{L^p}(K)$.

Let $f \in \widehat{L^p}(K) \subset \widehat{L^p}(\mathbb{R}^n)$ and $\phi \in C_0^\infty(\mathbb{R}^n)$. If $\operatorname{supp}\phi \subset \mathbb{R}^n \setminus K$, then $\langle \chi_K f, \phi\rangle = 0$ as in the previous paragraph, and so $\langle f - \chi_K f, \phi\rangle = 0$. Assume that $\operatorname{supp}\phi \subset K \setminus \partial K$. Let $g_j \in C_0^\infty(\mathbb{R}^n)$ be such that $\|g_j - \widehat{f}\|_p \to 0$ and $\widehat{f}_j = g_j$. Now $\chi_K f_j \to \chi_K f$ in $\widehat{L^p} \subset \mathscr{S}'$ by Lemma 6.4, so

$$\langle f - \chi_K f, \phi\rangle = \lim_{j\to\infty} \langle f_j - \chi_K f_j, \phi\rangle = \lim_{j\to\infty} (\langle f_j, \phi\rangle - \langle f_j, \chi_K \phi\rangle) = 0 \tag{115}$$

since $\chi_K \phi = \phi$. Hence $\operatorname{supp} f - \chi_K f \subset \partial K$, but then Lemma 5.4 implies that $f = \chi_K f$.

To establish the first claim of (114), let ϕ_j be the dyadic partition of unity. Then

$$\|\chi_K f\|_{\widehat{B^t_{p,r}}} = \left(\sum_{j=0}^{\ln_2 \sqrt{n}} \left(R_j^t \left\| \widehat{\chi_R \phi_j f} \right\|_p \right)^r \right)^{1/r} \tag{116}$$

since $K \subset \overline{B}(0, \sqrt{n})$. Now $\widehat{\chi_K \phi_j f} = C \widehat{\chi_K} * \widehat{\phi_j f}$ so Lemma 6.4 gives

$$\begin{aligned} \|\chi_K f\|_{\widehat{B^t_{p,r}}} &\le C \left(\sum_{j=0}^{\ln_2 \sqrt{n}} \left(R_j^t \left\| \widehat{\phi_j f} \right\|_p \right)^r \right)^{1/r} \\ &\le C \left(\sum_{j=0}^{\ln_2 \sqrt{n}} R_j^{(t-s)r} \right)^{1/r} \sup_{j\ge 0} R_j^s \left\| \widehat{\phi_j f} \right\|_p \end{aligned} \tag{117}$$

Noting the fact that $\ell^q \hookrightarrow \ell^\infty$ for all q, we get

$$\|\chi_K f\|_{\widehat{B^t_{p,r}}} \le C \|f\|_{\widehat{B^s_{p,q}}} . \tag{118}$$

For the last claim,

$$\begin{aligned} \|\chi_K f\|_{\widehat{L^p}} &= \left\| \widehat{\chi_K f} \right\|_p \le \sum_{j=0}^{\infty} \left\| \widehat{\chi_K \phi_j f} \right\|_p \le C \sum_{j=0}^{\ln_2 \sqrt{n}} \left\| \widehat{\phi_j f} \right\|_p \\ &\le C \left(\sum_{j=0}^{\ln_2 \sqrt{n}} R_j^{-sq'} \right)^{1/q'} \left(\sum_{j=0}^{\infty} \left(R_j^s \left\| \widehat{\phi_j f} \right\|_p \right)^q \right)^{1/q} = C \|f\|_{\widehat{B^s_{p,q}}} \end{aligned} \tag{119}$$

$\square$

Proposition 6.6. *Let $\varphi(x) \in C^\infty(\mathbb{R}^n)$ and χ_K be the characteristic function of a closed cube K. Then for all $1 < p < \infty$, $s \in \mathbb{R}$ and $0 < q \le \infty$*

$$Q := \chi_K \varphi \in \widehat{B^s_{p,q}} \cap \widehat{L^p}(K) \tag{120}$$

and multiplication by Q is a bounded operator from $\widehat{B^t_{p,r}}$ to $\widehat{B^s_{p,q}} \cap \widehat{L^p}(K)$, for any $t, s \in \mathbb{R}$, $0 < r \leq \infty$, i.e.

$$\|Qf\|_{\widehat{L^p}(K)} + \|Qf\|_{\widehat{B^s_{p,q}}} \leq C \|f\|_{\widehat{B^t_{p,r}}} . \tag{121}$$

Proof. We may assume that $\varphi \in C_0^\infty$. Then $\widehat{\varphi} \in L^1$ and $\widehat{\chi_K} \in L^p$ according to Lemma 6.4, so $Q \in \widehat{L^p}(R)$. Also,

$$\|Q\|_{\widehat{B^s_{p,q}}} = \left(\sum_{j=0}^{\infty} \left(R_j^s \left\| \widehat{\phi_j Q} \right\|_p \right)^q \right)^{1/q} \leq C \left(\sum_{j=0}^{j_K} \left(R_j^s \left\| \widehat{\phi_j} \right\|_1 \left\| \widehat{Q} \right\|_p \right)^q \right)^{1/q} \tag{122}$$

where the sum is finite because Q has compact support.

Let $f \in \widehat{B^t_{p,r}}$. Now, again because χ_K has bounded support, we have

$$\begin{aligned} \|Qf\|_{\widehat{L^p}} &= \left\| \widehat{\chi_K \varphi f} \right\|_p \leq \|\widehat{\varphi}\|_1 \left\| \widehat{\chi_K f} \right\|_p \leq C \|\widehat{\varphi}\|_1 \sum_{j=0}^{j_K} R_j^{-t} R_j^t \left\| \widehat{\chi_K \phi_j f} \right\|_p \\ &\leq C \|\widehat{\varphi}\|_1 \left(\sum_{j=0}^{j_K} R_j^{-tr'} \right)^{1/t'} \left(\sum_{j=0}^{\infty} \left(R_j^t \left\| \widehat{\phi_j f} \right\|_p \right)^r \right)^{1/r} \leq C \|f\|_{\widehat{B^t_{p,r}}} . \end{aligned} \tag{123}$$

The last claim follows from the fact that $\widehat{L^p}(K) \hookrightarrow \widehat{B^s_{p,q}}$ for all parameter values. ☐

Remark 6.7. Propositions 6.3 and 6.6, are used to estimate $\sum_j (G_\rho Q)^j f$ for a particular f and a Q with bounded support. If $\operatorname{supp} Q \subset K$, then we may as well just estimate

$$\sum_{j=0}^{\infty} (\chi_K G_\rho Q)^j f. \tag{124}$$

It is quite easy to see that $\chi_K G_\rho : \widehat{L^p}(K) \to \widehat{L^p}(K)$ using methods as in the proofs of Proposition 6.3 and Corollary 6.5. We decided to prove the results for the $\widehat{B^s_{p,q}}$ spaces at the same time because of their link to the spaces $\widehat{B^1_{2,1}}$ and $\widehat{B^{-1}_{2,\infty}}$ used in [1].

Proposition 6.8. *Let $K \subset \mathbb{R}^n$ be a closed cube, $1 < p < \infty$, $\varphi \in C^\infty$ and $Q = \chi_K \varphi$. Then*

$$f \mapsto \chi_R G_\rho Q f : \widehat{L^p}(K) \to \widehat{L^p}(K) \tag{125}$$

$$f \mapsto G_\rho Q f : \widehat{B^{-1}_{p,\infty}} \to \widehat{B^{-1}_{p,\infty}} \tag{126}$$

for all $\rho \in \mathbb{C}^n$, $\rho \cdot \rho = 0$, with norm estimates

$$\|\chi_R G_\rho Q f\|_{\widehat{L^p}(K)} \leq \frac{C}{|\rho|} \|f\|_{\widehat{L^p}(K)} , \tag{127}$$

$$\|G_\rho Q f\|_{\widehat{B^{-1}_{p,\infty}}} \leq \frac{C}{|\rho|} \|f\|_{\widehat{B^{-1}_{p,\infty}}} , \tag{128}$$

where C does not depend on f or $|\rho|$.

Proof. We may assume that $\varphi \in C_0^\infty$. Then multiplying by it is a bounded operator in $\widehat{L^p}(R)$ and $\widehat{B^s_{p,q}}$. The claim follows from Corollary 6.5 and Proposition 6.3 then. $\square$

We are now ready to prove the existence of the CGO solutions.

Theorem 6.9. *Let $K \subset \mathbb{R}^n$ be a closed cube, $\varphi \in C^\infty(\mathbb{R}^n)$ and $Q = \chi_K \varphi$. Let $1 < p < \infty$, $\rho \in \mathbb{C}^n$, $\rho \cdot \rho = 0$ and $|\rho| > 2C_{\varphi,K,p}$. Then the equation $(\Delta - Q)w = 0$ has a solution*

$$w = e^{x\cdot\rho}(1 + \psi) \tag{129}$$

in the domains $\operatorname{int} K$*,* $\mathbb{R}^n$*, with $\psi \in \widehat{L^p}(K)$, $\psi \in \widehat{B^{-1}_{p,\infty}}$, respectively. In both cases*

$$\|\psi\| \leq \frac{2C_{\varphi,K,p}}{|\rho|}. \tag{130}$$

Proof. Recalling (94), $\psi = \sum_{k=1}^\infty (G_\rho Q)^k 1$, but constant functions are not in the space $\widehat{B^{-1}_{p,\infty}}$. Take $f \in C_0^\infty$ such that $f \equiv 1$ on K. Then $f \in \widehat{B^{-1}_{p,\infty}}$ and $\psi = \sum_{k=1}^\infty (G_\rho Q)^k f$. Proposition 6.8 implies

$$\|\psi\|_{\widehat{B^{-1}_{p,\infty}}} \leq \sum_{k=1}^\infty \left(\frac{C_{\varphi,K,p}}{|\rho|}\right)^k \|f\|_{\widehat{B^{-1}_{p,\infty}}} \leq \frac{C_{\varphi,K,p}}{|\rho|} \frac{1}{1 - \frac{C_{\varphi,K,p}}{|\rho|}}, \tag{131}$$

which is at most $2C_{\varphi,K,p}\,|\rho|^{-1}$ as long as $|\rho| > 2C_{\varphi,K,p}$. The space $\widehat{B^{-1}_{p,\infty}}$ is complete so we have existence and the norm estimate.

Consider the equation in the interior of K now. There $\chi_K \equiv 1$, so $\psi = \sum_{k=1}^\infty (\chi_K G_\rho Q)^k \chi_K$ satisfies the Faddeev equation inside K. We have $\chi_K \in \widehat{L^p}(K)$ by Lemma 6.4, so Proposition 6.8 gives the last claim like in the previous paragraph. $\square$

Proof of theorem 3.3. We have $m = \chi_K \varphi(x)$. Let $Q = -k^2(f - \varphi)\chi_K$, where $f \in C_0^\infty$ is identically one in K. Let $2 \leq p < \infty$ and let $w = e^{x\cdot\rho}(1+\psi) \in \widehat{L^{p'}}(K)$, $1/p + 1/p' = 1$, be the CGO solution from Theorem 6.9. Now

$$(\Delta + k^2(1 - m))w = 0 \quad \text{inside } K. \tag{132}$$

Young's inequality gives

$$\|\psi\|_{L^p(K)} \leq \|\widehat{\psi}\|_{p'} \leq \frac{C}{|\rho|}. \tag{133}$$

$\square$

Remark 6.10. We could also use the solutions $w \in \widehat{B^{-1}_{p,\infty}}$ satisfying

$$(\Delta - Q)w = 0 \quad \text{in } \mathbb{R}^n. \tag{134}$$

Then for any compact set A we have

$$\|\psi\|_{L^p(A)} \leq \sum_{j=0}^{j_A} \|\psi_j\|_{p'} = \|\psi\|_{\widehat{B^{-1}_{p',\infty}}} \sum_{j=0}^{j_A} R_j \leq \frac{C}{|\rho|}, \tag{135}$$

where $j_A < \infty$ is large enough that A is contained in the ball of radius $2^{j_A - 1}$. Note that we haven't proven existence for $Q = -k^2(1-m)$ since 1 is not of the form $\chi_K \varphi(x)$, but we may take a big rectangle $\widetilde{K} \supset A$ and consider $Q = -k^2(\chi_{\widetilde{K}} - \chi_K \varphi)$, which will work well with our propositions.

Remark 6.11. Let A be a compact set. It is not hard to see that all the results of this section work for potentials $Q \in \widehat{L^p}(A)$ that are also Fourier-multipliers for L^p. Hence we could have taken an arbitrary polytope instead of K.

7 Proof of theorem 3.5

In this section we will use notation that is different from that used in the rest of the paper. We will be dealing with polynomials in n complex variables. We will use a notation for what is sometimes called array operations, as well as standard multi-index notation. Specifically, for $\eta \in \mathbb{C}^n$, with α a multi-index,

$$\eta^\alpha = \eta_1^{\alpha_1} \cdots \eta_n^{\alpha_n} \tag{136}$$

is a scalar, but

$$\eta^2 = (\eta_1^2, \dots, \eta_n^2) \tag{137}$$

is the vector with squared components. Similarly,

$$\frac{1}{\eta} = \left(\frac{1}{\eta_1}, \dots, \frac{1}{\eta_n}\right), \tag{138}$$

$$1 = (1, \dots, 1), \tag{139}$$

$$\frac{x}{\eta} = \left(\frac{x_1}{\eta_1}, \dots, \frac{x_n}{\eta_n}\right). \tag{140}$$

Also, $\sigma_k(\eta)$ denotes the k'th elementary symmetric function of $(\eta_1, \dots, \eta_n)$,

$$\sigma_k(\eta) = \sum_{1 \leq j_1 < j_2 < \cdots < j_k \leq n} \eta_{j_1} \eta_{j_1} \cdots \eta_{j_n}. \tag{141}$$

Below we will only use

$$\sigma_n(\eta) = \prod_{i=1}^{n} \eta_i, \tag{142}$$

$$\sigma_{n-1}(\eta) = \sum \prod_{j \neq i} \eta_j. \tag{143}$$

The hat $\widehat{}$ indicates that an index does not occur, so that

$$\eta_{\hat{\imath}} = (\eta_1, \ldots, \eta_{i-1}, \eta_{i+1}, \ldots, \eta_n) \tag{144}$$

means the $n-1$-dimensional vector that omits the i'th component of η. When P denotes a polynomial, we will often use the notation $P_{\hat{\imath}}$ and $P(\eta_{\hat{\imath}})$ interchangeably to denote a polynomial which does not depend on the i'th variable.

The Laplace transform of a degree N homogeneous polynomial $P(x)$ is given by

$$\widehat{P}(\rho) = \int_{x>0} e^{-\rho\cdot x} P(x) dx \tag{145}$$

where $x > 0$ means $x_i > 0$ for each component. Letting $y_i = \rho_i x_i$, with ρ real and $\rho > 0$ (for the moment) we have

$$\widehat{P}(\rho) = \int_{y>0} e^{-1\cdot y} P\left(\frac{y}{\rho}\right) \sigma_n \left(\frac{1}{\rho}\right) dy. \tag{146}$$

If $P = \sum_{|\alpha|=N} p_\alpha x^\alpha$, then

$$\widehat{P}(\rho) = \sum_{|\alpha|=N} p_\alpha \frac{1}{\rho^{\alpha+1}} \int_{y>0} e^{-1\cdot y} y^\alpha dy = \sum_{|\alpha|=N} p_\alpha \left(\frac{1}{\rho}\right)^{\alpha+1} \alpha! \tag{147}$$

so

$$\widehat{P}(\rho) = Q^{N+n}\left(\frac{1}{\rho}\right) \tag{148}$$

for a homogeneous polynomial $Q = Q^{N+n}$ of degree $N+n$.

The main assertion of Theorem 3.5 is that

$$Q\Big|_{\rho\cdot\rho=0} \not\equiv 0 \tag{149}$$

Because Q is a polynomial in $\frac{1}{\rho}$, we will work with the new variables $\eta = \frac{1}{\rho}$. After that we may continue it analytically to the whole $\mathbb{C}^n$. In terms of η

$$\rho\cdot\rho = \frac{1}{\eta}\cdot\frac{1}{\eta} = \frac{\sigma_{n-1}(\eta^2)}{\sigma_n(\eta^2)} = \frac{\sigma_{n-1}(\eta^2)}{(\sigma_n(\eta))^2}. \tag{150}$$

If P is harmonic, $Q(\eta) = \widehat{P}(\frac{1}{\eta})$ has an additional representation.

Lemma 7.1. *If P is harmonic and homogeneous, then*

$$Q(\eta) = \widehat{P}(\frac{1}{\eta}) = \frac{\sigma_n(\eta)}{\sigma_{n-1}(\eta^2)} \sum_{i=1}^{n} (P_{\hat{\imath}} + \eta_i Q_{\hat{\imath}}) \tag{151}$$

where $P_{\hat{\imath}}$ and $Q_{\hat{\imath}}$ are homogeneous polynomials of degree $N+2n-2$, $N+2n-3$, respectively, which do not depend on η_i.

Proof. Assume that $\Re\rho_i > 0$ and prove the formula (151). Then continue both sides of the equation analytically to the whole $\mathbb{C}^n$. Assuming $\rho \cdot \rho \neq 0$ for a moment, we obtain, by integrating by parts and recalling that P is harmonic, that

$$\begin{aligned}
\widehat{P}(\rho) &= \int_{x>0} \frac{\Delta e^{-\rho\cdot x}}{\rho\cdot\rho} P(x)dx \\
&= \frac{1}{\rho\cdot\rho}\sum_{i=1}^{n}\left(\int_{\substack{x_i=0\\ x_{\hat{\imath}}>0}} e^{-\rho\cdot x}\left(\rho_i P + \frac{\partial}{\partial x_i}P\right) + \int_{x>0} e^{-\rho\cdot x}\Delta P\right) \\
&= \frac{1}{\rho\cdot\rho}\sum_{i=1}^{n}\left(\rho_i \int_{x_{\hat{\imath}}>0} e^{-\rho_{\hat{\imath}}\cdot x_{\hat{\imath}}} P_{|x_i=0} + \int_{x_{\hat{\imath}}>0} e^{-\rho_{\hat{\imath}}\cdot x_{\hat{\imath}}} \frac{\partial}{\partial x_i} P_{|x_i=0}\right) \\
&= \frac{1}{\rho\cdot\rho}\sum_{i=1}^{n}\left(\rho_i P_i(\frac{1}{\rho_{\hat{\imath}}}) + Q_i(\frac{1}{\rho_{\hat{\imath}}})\right) \qquad (152)
\end{aligned}$$

If we rewrite this in terms of η, we obtain by using (150)

$$\begin{aligned}
Q(\eta) &= \frac{(\sigma_n(\eta))^2}{\sigma_{n-1}(\eta^2)}\sum_{i=1}^{n}\left(\frac{1}{\eta_i}P_i(\eta_{\hat{\imath}}) + Q_i(\eta_{\hat{\imath}})\right) \\
&= \frac{\sigma_n(\eta)}{\sigma_{n-1}(\eta^2)}\sum_{i=1}^{n}\left(\frac{\sigma_n(\eta)}{\eta_i}P_i(\eta_{\hat{\imath}}) + \sigma_n(\eta)Q_i(\eta_{\hat{\imath}})\right) \\
&= \frac{\sigma_n(\eta)}{\sigma_{n-1}(\eta^2)}\sum_{i=1}^{n}\left(\widetilde{P}_i(\eta_{\hat{\imath}}) + \eta_i\widetilde{Q}_i(\eta_{\hat{\imath}})\right), \qquad (153)
\end{aligned}$$

where

$$\widetilde{P}_i(\eta_{\hat{\imath}}) = \sigma_{n-1}(\eta_{\hat{\imath}})P_i(\eta_{\hat{\imath}}) \qquad (154)$$

and

$$\widetilde{Q}_i(\eta_{\hat{\imath}}) = \sigma_{n-1}(\eta_{\hat{\imath}})Q_i(\eta_{\hat{\imath}}). \qquad (155)$$

□

The proof of Theorem 3.5 is now reduced to the following two propositions.

Proposition 7.2. *Let P be harmonic and homogeneous. If $n \geq 3$ and $\widehat{P}(\rho)$ vanishes identically in a nonempty open subset of $\{\rho\cdot\rho = 0, \Re\rho > 0\}$, or $n = 2$ and $\widehat{P}(\rho)$ vanishes identically on $\{\rho\cdot\rho = 0, \Re\rho > 0\}$, then $\sigma_{n-1}(\eta^2)$ divides the polynomial $Q(\eta) = \widehat{P}(\frac{1}{\eta})$.*

Proposition 7.3. *$\sigma_{n-1}(\eta^2)$ cannot divide any polynomial Q of the form* (151).

Both propositions rely on the following lemma.

Lemma 7.4. *If $\eta \in \mathbb{C}^n$ and $\eta \geq 3$, then $\sigma_{n-1}(\eta^2)$ is an irreducible polynomial. When $n = 2$, then $\sigma_{n-1}(\eta^2) = \eta_1^2 + \eta_2^2 = (\eta_1 - i\eta_2)(\eta_1 + i\eta_2)$.*

Proof. We need only to consider the case $n \geq 3$. We will prove this lemma by induction, making use of the identity

$$\sigma_{n-1}(\eta^2) = \eta_1^2 \sigma_{n-2}(\eta_{\hat{1}}^2) + \sigma_{n-1}(\eta_{\hat{1}}^2) \tag{156}$$

for $\eta \in \mathbb{C}^n$. If $\sigma_{n-1}(\eta^2)$ factors, and one factor does not depend on η_1, then we must have

$$\eta_1^2 \sigma_{n-2}(\eta_{\hat{1}}^2) + \sigma_{n-1}(\eta_{\hat{1}}^2) = p_{\hat{1}}(\eta_1^2 q_{\hat{1}} + r_{\hat{1}}) \tag{157}$$

where $p_{\hat{1}}, q_{\hat{1}}$, and $r_{\hat{1}}$ are non-constant polynomials independent of η_1. Equating coefficients of η_1^2 gives

$$\sigma_{n-2}(\eta_{\hat{1}}^2) = p_{\hat{1}} q_{\hat{1}}, \tag{158}$$

which contradicts the induction hypothesis because $\sigma_{n-2}(\eta_{\hat{1}}^2) = \sigma_{m-1}(\xi^2)$ with $m = n-1$ and $\xi = \eta_{\hat{1}} \in \mathbb{C}^m$.

If both factors depend on η_1, i.e.

$$\eta_1^2 \sigma_{n-2}(\eta_{\hat{1}})^2 + \sigma_{n-1}(\eta_{\hat{1}}^2) = (\eta_1 p_{\hat{1}} + q_{\hat{1}})(\eta_1 r_{\hat{1}} + s_{\hat{1}}), \tag{159}$$

equating coefficients of η_1^2 again gives

$$\sigma_{n-2}(\eta_{\hat{1}}) = p_{\hat{1}} q_{\hat{1}} \tag{160}$$

which also contradicts the induction hypothesis.

To finish the induction argument we still need to prove the case $n = 3$. Then

$$\sigma_2(\eta^2) = \eta_1^2(\eta_2 + i\eta_3)(\eta_2 - i\eta_3) + \eta_2^2\eta_3^2. \tag{161}$$

If $\sigma_2(\eta^2) = p_{\hat{1}}(\eta_1^2 q_{\hat{1}} + r_{\hat{1}})$, then, equating the coefficients of η_1^2, we see again that $p_{\hat{1}}$ must divide $(\eta_2 + i\eta_3)(\eta_2 - i\eta_3)$ and $\eta_2^2\eta_3^2$. This is impossible because the two have prime factorizations without common factors. If, on the other hand,

$$\sigma_2(\eta^2) = (\eta_1 p_{\hat{1}} + q_{\hat{1}})(\eta_1 r_{\hat{1}} + s_{\hat{1}}), \tag{162}$$

then

$$p_{\hat{1}} q_{\hat{1}} = (\eta_2 + i\eta_3)(\eta_2 - i\eta_3) \tag{163}$$

which implies that $p_{\hat{1}}$ must be either $(\eta_2 + i\eta_3)$ or $(\eta_2 - i\eta_3)$ multiplied by a constant and $q_{\hat{1}}$ must be the other. Also

$$q_{\hat{1}} r_{\hat{1}} = -p_{\hat{1}} s_{\hat{1}} \tag{164}$$

so that $p_{\hat{1}}$ must divide $r_{\hat{1}}$ because it doesn't divide $q_{\hat{1}}$. However

$$r_{\hat{1}} s_{\hat{1}} = \eta_2^2 \eta_3^2 \tag{165}$$

does not have $(\eta_2 \pm i\eta_3)$ as a factor, so this is impossible. $\square$

Proof of Proposition 7.2. We will first show that $Q(\eta)$ vanishes on the set $\{\eta \mid \sigma_{n-1}(\eta^2) = 0\}$. Recall that

$$\frac{1}{\eta} \cdot \frac{1}{\eta} = \frac{\sigma_{n-1}(\eta^2)}{(\sigma_n(\eta))^2}. \tag{166}$$

Let

$$V = \{\eta \mid \sigma_{n-1}(\eta^2) = 0\}. \tag{167}$$

If $n \geq 3$ then $\sigma_{n-1}(\eta^2)$ is irreducible by Lemma 7.4, so V is an irreducible variety. If $n = 2$, then $V = V_1 \cup V_2$ for two irreducible varieties V_1 and V_2.

The assumptions imply that Q vanishes on a nonempty open subset ($n \geq 3$), or the whole ($n = 2$) of

$$\{\sigma_{n-1}(\eta^2) = 0, \Re\eta > 0\} \setminus \{\sigma_n(\eta) = 0\} \subset V. \tag{168}$$

If a polynomial vanishes on a nonempty open subset of an irreducible variety, then it vanishes on the whole variety ([12, p. 91] or [16]). Hence Q vanishes on the whole V. Hilbert's Nullstellensatz implies that Q has all the irreducible factors of $\sigma_{n-1}(\eta^2)$, but the latter is square-free. It must be a factor of Q. $\square$

Proof of Proposition 7.3. Q is a polynomial which, according to (151), has the form

$$Q(\eta) = \frac{\sigma_n(\eta) \sum (P_{\hat{\imath}} + \eta_i Q_{\hat{\imath}})}{\sigma_{n-1}(\eta^2)}.$$

If $\sigma_{n-1}(\eta^2)$ divides $Q(\eta)$, then its square $\big(\sigma_{n-1}(\eta^2)\big)^2$ divides the numerator. As $\sigma_n(\eta) = \prod \eta_i$ has a unique prime factorization that does include the irreducible $\sigma_{n-1}(\eta^2)$ (or its factors in the case $n = 2$), this is only possible if

$$\sum (P_{\hat{\imath}} + \eta_i Q_{\hat{\imath}}) = \big(\sigma_{n-1}(\eta^2)\big)^2 R(\eta) \tag{169}$$

for some homogeneous polynomial $R(\eta) = \sum\limits_{|\alpha|=N} c_\alpha \eta^\alpha$. The proof will be completed once we show that (169) is impossible.

Setting $\xi_i = \prod_{k \neq i} \eta_k^2$ in the identity

$$\Big(\sum \xi_i\Big)^2 = \sum \xi_i^2 + 2 \sum_{i \neq j} \xi_i \xi_j. \tag{170}$$

yields

$$\big(\sigma_{n-1}(\eta^2)\big)^2 = \sigma_{n-1}(\eta^4) + 2\sigma_{n-2}(\eta^2)\sigma_n(\eta^2). \tag{171}$$

We continue by observing that the right hand side of (169) is equal to

$$\begin{aligned} &\sum_{|\alpha|=N} c_\alpha \big(\sigma_{n-1}(\eta^4)\eta^\alpha + 2\sigma_{n-2}(\eta^2)\eta^{2+\alpha}\big) \\ &\qquad = \sum_{|\beta|=N+4(n-1)} \Big(\sum_{j=1}^{n} c_{(\beta-4)+4e_j} + \sum_{1 \leq j < k \leq n} 2c_{(\beta-4)+2e_j+2e_k}\Big)\eta^\beta, \end{aligned} \tag{172}$$

where we have defined that $c_\beta = 0$ if any component of β is negative. Note that we used the notation $\beta + c = (\beta_1 + c, \ldots, \beta_n + c)$ for a vector β and a scalar c.

Next, we will compare the coefficients of certain terms of $\sum(P_{\hat{\imath}} + \eta_i Q_{\hat{\imath}})$ and the right hand side of (172). When all the exponents are greater than two, the term does not appear in $\sum(P_{\hat{\imath}} + \eta_i Q_{\hat{\imath}})$. In each row of the following table let $\widetilde{\beta} \in \mathbb{N}^{n-2}$ with $\left|\widetilde{\beta}\right|$ given by the first column. The second column tells which term we are interested in. Then the corresponding coefficient in $\sum(P_{\hat{\imath}} + \eta_i Q_{\hat{\imath}})$ and the right hand side of (172) are given by the two remaining columns.

$\left\lvert\widetilde{\beta}\right\rvert$	Term	$\sum(P_{\hat{\imath}} + \eta_i Q_{\hat{\imath}})$	RHS
N	$\eta^{(2,2,\widetilde{\beta}+4)}$	0	$2c_{0,0,\widetilde{\beta}}$
$N-1$	$\eta^{(3,2,\widetilde{\beta}+4)}$	0	$2c_{1,0,\widetilde{\beta}}$
$N-1$	$\eta^{(2,3,\widetilde{\beta}+4)}$	0	$2c_{0,1,\widetilde{\beta}}$
$N-2$	$\eta^{(3,3,\widetilde{\beta}+4)}$	0	$2c_{1,1,\widetilde{\beta}}$

(173)

The second last column is full of zeros because $\sum(P_{\hat{\imath}} + \eta_i Q_{\hat{\imath}})$ does not have terms divisible by $\sigma_n(\eta^2)$. The last column looks like that because in (172) the terms of $\sigma_{n-1}(\eta^4)\eta^\alpha$ have all but one exponent at least 4, and we only get the term with $j = 1$, $k = 2$ from the second sum.

Consider the coefficient of the term η^β with $\beta_1, \ldots, \beta_n \geq 2$. This term does not appear on the left hand side of (169), so we get the array of equations

$$\sum_{j=1}^{n} c_{(\beta-4)+4e_j} + \sum_{1\leq j<k\leq d} 2c_{(\beta-4)+2e_j+2e_k} = 0, \qquad \beta_1, \ldots, \beta_n \geq 2. \tag{174}$$

In low dimensions this is just

n	equation
2	$c_{\beta_1-4,\beta_2} + c_{\beta_1,\beta_2-4} + 2c_{\beta_1-2,\beta_2-2} = 0$
3	$c_{\beta_1,\beta_2-4,\beta_3-4} + c_{\beta_1-4,\beta_2,\beta_3-4} + c_{\beta_1-4,\beta_2-4,\beta_3} + 2c_{\beta_1-2,\beta_2-2,\beta_3-4} + 2c_{\beta_1-2,\beta_2-4,\beta_3-2} + 2c_{\beta_1-4,\beta_2-2,\beta_3-2} = 0$

(175)

Note that all of our equations have jumps of even size in the indices. We also have the initial values (173) for all of the four cases $(\beta_1, \beta_2) \pmod 2$. Thus it is enough to consider the case where both β_1 and β_2 are even.

The equation array can be visualized as in figure 1 when $n = 3$. The proof works in the exact same way in other dimensions. We write $\widetilde{\beta} = (\beta_3, \ldots, \beta_n)$ for $\beta \in \mathbb{N}^n$. The nodes in the bigger triangle correspond to the values of c_β. We superimpose the small triangle over the big one. Then its location corresponds to one of the equations in (174). Each node outside the big triangle would correspond to some c_β with β having a negative index. The small triangle may be partly outside the bigger one. This is allowed as long as the triangles have at least two common nodes. This follows from the condition that $\beta_i \geq 2$ for all i in (174).

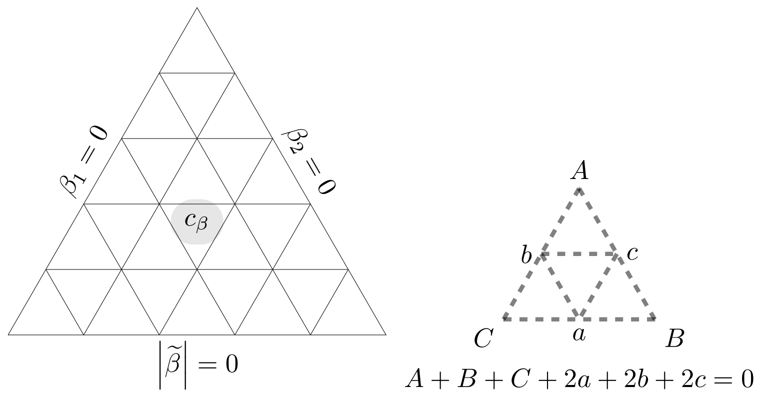
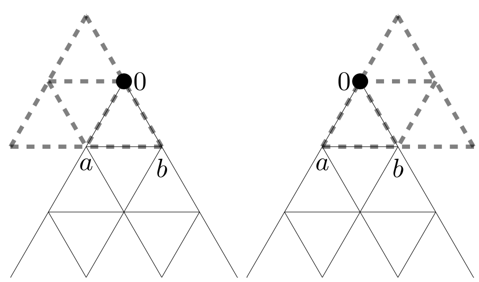


Figure 1: The coefficients c_β and their equations (174)

We proceed by induction, proving that each row in the triangle is full of zeros. Figures 2 and 3 show the idea. We start from the top. When the previous rows have been shown to contain only zeros, we are reduced to the 2-dimensional case. Then we move the small triangle to the right one step at a time. On each step, we solve the value of the rightmost node based on the values of the middle and leftmost one.



$$\left.\begin{array}{l} 2a + b = 0 \\ a + 2b = 0 \end{array}\right\} \Rightarrow a = b = 0$$

Figure 2: The first induction step

The value of the top node is given by $c_{0,0,\widetilde{\beta}} = 0$ in (173). Each row can be reduced to the 2-dimensional case as follows: Assume that we have proved that $c_\alpha = 0$ whenever $\alpha_1 + \alpha_2 \leq k$. Then move the small triangle down one step, or equivalently, choose $(\beta_1, \beta_2, \widetilde{\beta}) = (\alpha_1 + 2, \alpha_2 + 2, \widetilde{\alpha})$ with $\alpha_1 + \alpha_2 = k + 2$ in (174). The coefficients c_α with $\alpha_1 + \alpha_2 \in \{k, k-2\}$ vanish by the induction assumption, so we are left with

$$c_{\alpha_1+2,\alpha_2-2,\widetilde{\alpha}-4} + c_{\alpha_1-2,\alpha_2+2,\widetilde{\alpha}-4} + 2c_{\alpha_1,\alpha_2,\widetilde{\alpha}-4} = 0. \qquad (176)$$

This is valid as long as $\alpha_1, \alpha_2 \geq 0$, $\widetilde{\alpha}_i \geq 2$ and $\alpha_1 + \alpha_2 = k + 2$.

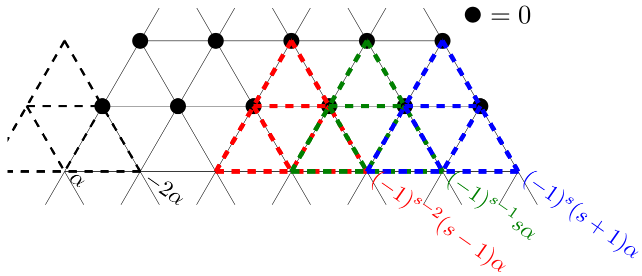


Figure 3: Moving the triangle to the right on an arbitrary induction step

Next, write $A = c_{0,k+2,\widetilde{\alpha}-4}$ for the leftmost node. Let $\alpha_1 = 0$, $\alpha_2 = k+2$ in (176). Then

$$c_{2,k,\widetilde{\alpha}-4} + 2c_{0,k+2,\widetilde{\alpha}-4} = 0 \quad \Longrightarrow \quad c_{2,k,\widetilde{\alpha}-4} = -2A. \tag{177}$$

Next, move the triangle to the right, or let (α_1, α_2) go through all the pairs in $\{(2,k), (4,k-2), \ldots, (k,2)\}$. A simple induction gives us

$$c_{2s,k-2s,\widetilde{\alpha}-4} = (-1)^s(s+1)A \tag{178}$$

for $0 \leq s \leq \frac{k}{2} + 1$. Then finally, put the triangle half-way outside on the right, or choose $\alpha_1 = k+2$, $\alpha_2 = 0$ to get

$$\begin{aligned} c_{k,2,\widetilde{\alpha}-4} + 2c_{k+2,0,\widetilde{\alpha}-4} = 0 \\ \Leftrightarrow (-1)^{k/2}\big(\tfrac{k}{2}+1\big)A + 2(-1)^{k/2+1}\big(\tfrac{k}{2}+2\big)A = 0 \\ \Leftrightarrow A = 0 \end{aligned} \tag{179}$$

Hence $c_\alpha = 0$ for all $\alpha \in \mathbb{N}^n$ with $\alpha_1 + \alpha_2 \leq k+2$. Induction gives the claim.

□

References

[1] Shmuel Agmon and Lars Hörmander. Asymptotic properties of solutions of differential equations with simple characteristics. *J. Analyse Math.*, 30:1–38, 1976.

[2] Jöran Bergh and Jörgen Löfström. *Interpolation spaces. An introduction.* Springer-Verlag, Berlin, 1976. Grundlehren der Mathematischen Wissenschaften, No. 223.

[3] Fioralba Cakoni, Drossos Gintides, and Houssem Haddar. The existence of an infinite discrete set of transmission eigenvalues. *SIAM J. Math. Anal.*, 42(1):237–255, 2010.

[4] David Colton and Andreas Kirsch. A simple method for solving inverse scattering problems in the resonance region. *Inverse Problems*, 12(4):383–393, 1996.

[5] David Colton and Rainer Kress. *Inverse acoustic and electromagnetic scattering theory*, volume 93 of *Applied Mathematical Sciences*. Springer-Verlag, Berlin, 1992.

[6] David Colton and Peter Monk. The inverse scattering problem for time-harmonic acoustic waves in an inhomogeneous medium. *Quart. J. Mech. Appl. Math*, 41(1):97–125, 1988.

[7] David Colton, Lassi Päivärinta, and John Sylvester. The interior transmission problem. *Inverse Probl. Imaging*, 1(1):13–28, 2007.

[8] Michael Hitrik, Katsiaryna Krupchyk, Petri Ola, and Lassi Päivärinta. Transmission eigenvalues for operators with constant coefficients. *SIAM J. Math. Anal.*, 42(6):2965–2986, 2010.

[9] Lars Hörmander. *The analysis of linear partial differential operators. I.* Classics in Mathematics. Springer-Verlag, Berlin, 2003. Distribution theory and Fourier analysis, Reprint of the second (1990) edition.

[10] Andreas Kirsch and Natalia Grinberg. *The factorization method for inverse problems*, volume 36 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, Oxford, 2008.

[11] Lassi Päivärinta and John Sylvester. Transmission eigenvalues. *SIAM J. Math. Anal.*, 40(2):738–753, 2008.

[12] Joseph Fels Ritt. *Differential Equations from the Algebraic Standpoint.* American mathematical society. Colloquium publications, 1932.

[13] Alberto Ruiz. Harmonic Analysis and Inverse Problems. lecture notes, http://www.uam.es/gruposinv/inversos/publicaciones/Inverseproblems.pdf, accessed 2012.

[14] Elias Menachem Stein. *Singular integrals and differentiability properties of functions*. Princeton Mathematical Series, No. 30. Princeton University Press, Princeton, N.J., 1970.

[15] Hans Triebel. *Theory of function spaces*, volume 78 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, 1983.

[16] Bartel Leendert van der Waerden. Zur algebraischen Geometrie. III. *Math. Ann.*, 108(1):694–698, 1933.